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EDGE SETS CONTAINED IN CIRCUITS

BY

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ABSTRACT

A graph G with n vertices has property $p(r, s)$ if G contains a path of length r and if every such path is contained in a circuit of length s. G. A. Dirac and C. Thomassen [Math. Ann. **203** (1973), 65–75] determined graphs with property $p(r, r+1)$. We determine the least number of edges in a graph G in order to insure that G has property $p(r, s)$, we determine the least number of edges possible in a connected graph with property $p(r, s)$ for $r = 1$ and all s, for $r = k$ and $s = k+2$ when $k = 2, 3, 4$, and we give bounds in other cases. Some resulting extremal graphs are determined. We also consider a generalization of property $p(2, s)$ in which it is required that each pair of edges is contained in a circuit of length s. Some cases of this last property have been treated previously by U. S. R. Murty [in *Proof Techniques in Graph Theory*, ed. F. Harary, Academic Press, New York, 1969, pp. 111–118].

1. Introduction

By a theorem of G. A. Dirac and C. Thomassen [5, theor. 1], if G is a connected graph which contains a path of length r and every such path in G is contained in a circuit of length $r+1$, then G is either a complete graph, a circuit, or a complete bipartite graph. A result of G. Chartrand and H. V. Kronk [4] shows that this is also a characterization of graphs with n vertices such that every path of length $n-2$ is contained in a circuit of length n (a Hamiltonian circuit), where the bipartite case occurs only if n is even and G is $K_{n/2,n/2}$. In this paper we place these theorems in the framework of a general family of problems and treat some extremal problems thus arising. Let us say that a graph G with n vertices has property $p(r, s)$ (respectively, $p(r, \leqq s)$), $1 \leqq r < s \leqq n$, if G contains a path of length r and if every such path is contained in a circuit of length s (respectively, less than or equal to s). Then the above-mentioned theorems determine graphs with property $p(r, r+1)$ and graphs with property $p(n-2, n)$.

We can view property $p(r, s)$ as a generalized undirected version of a bypass as described for directed graphs in [3]. That is, if P is a path of length r from vertex

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a to vertex b in a graph G with property $p(r, s)$, then P is contained in a circuit C of length s. The path of length $s - r$ from a to b which is contained in C, but distinct from P, can be thought of as a bypass of P from a to b. However, the translation of the definition of property $p(r, s)$ to directed graphs results in: a directed graph D has property $p(r, s)$, $1 \leqq r < s \leqq n$, if D contains a directed path of length r and if every such path is contained in a directed circuit of length s. B. Alspach proved (see [1]) that every regular tournament of order n has property $p(1, s)$, $3 \leqq s \leqq n$, and there are almost regular tournaments of even order n without property $p(1, s)$. The vertex version for strong tournaments, i.e., in every strong tournament of order n each vertex is contained in a directed circuit of length s, $3 \leqq s \leqq n$, is well known [8].

We also note a result by H. V. Kronk [7, theor. 2] concerning r-path Hamiltonian graphs, i.e., graphs with n vertices in which every path of length not exceeding r, $1 \leqq r \leq n - 2$, is contained in a circuit of length n. Namely, if $1 \leqq r \leqq n - 3$ and if G is a graph on n vertices and at least $\binom{n-1}{2} + r + 2$ edges, then G is r-path Hamiltonian. This implies the second case in Corollary 2.4 below. Graphs which are $(n-2)$-path Hamiltonian were called randomly Hamiltonian in [4] and are exactly those graphs with property $p(n-2, n)$.

In Sections 2–5 we are concerned with two extremal problems arising from property $p(r, s)$. For integers $1 \leqq r < s \leqq n$, determine the smallest integer $m_1(n; r, s) = m_1$ so that every graph with n vertices and m_1 edges has property $p(r, s)$, and determine the smallest integer $m_2(n; r, s) = m_2$ so that there exists a connected graph with n vertices and m_2 edges which has property $p(r, s)$. The value of $m_1(n; r, s)$ is determined in Section 2, and some extremal graphs are described. The determination of $m_2(n; r, s)$ is more involved, but in Section 3 we determine $m_2(n; 1, s)$ for all $2 \leqq s \leqq n$ and describe the extremal graphs when $n \equiv 1 \pmod{s-1}$, and in Section 4 we determine $m_2(n; 2, s)$ for $s = 3, 4$, describe the extremal graphs and determine $m_2(n; 3, 5)$. Bounds in certain other cases and the value of $m_2(n; 4, 6)$ are obtained in Section 5. In Section 6 we study a generalization of property $p(2, s)$. Special cases of this were studied by U. S. R. Murty and B. Bollobás (see [9]).

Familiarity with the basic notions of graph theory is assumed. Our terminology and notation is, in the most part, that of F. Harary [6]. Exceptions are that we use vertex and edge instead of point and line, and we often subscript graphical parameters with graphs in order to emphasize the graph in which the parameter is considered (e.g., $d_G(x, y)$ denotes the distance between vertices x and y in graph G).

2. The function $m_1(n; r, s)$

DEFINITION 2.1. Let r and n be positive integers, $r \leqq n-3$. Denote by H_r the graph obtained from K_{r+5} by deleting the edges of a circuit of length three. Denote by $H_{n,r}$ the graph obtained from K_{n-1} by adjoining a new vertex x which is adjacent to any other $r+1$ vertices of K_{n-1}.

REMARK 2.2. Any path in H_r of length r using only vertices of degree $r+4$ is contained in no circuit of length $r+5$. Any path in $H_{n,r}$ of length r using only vertices adjacent to x is contained in no circuit of length n. However, any other path of length r in either H_r or $H_{n,r}$ is in a hamiltonian circuit. Note, also, that a path in $H_{n,r}$ of length $r+1$ from x to a vertex adjacent to x which uses all vertices adjacent to x is contained in no circuit of length greater than $r+2$.

Clearly, $H_{n,n-3}$ has none of the properties $p(r, r+1)$ $(1 \leqq r \leqq n-1)$, $p(n-2, n)$. Hence $m_1(n; r, r+1) = m_1(n; n-2, n) = \binom{n}{2}$. Combining this with our first theorem below we obtain the value of $m_1(n; r, s)$.

THEOREM 2.3. *Let r and n be integers, $1 \leqq r \leqq n-3$. If G is a graph with n vertices and at least $\binom{n-1}{2} + r + 1$ edges, then G has property $p(r, s)$ for every $s = r+2, r+3, \cdots, n-1$. If, furthermore, $G \neq H_r$ and $G \neq H_{n,r}$, then G has property $p(r, n)$.*

Note that Theorem 2.3, in particular, implies the well-known result that every graph G with n vertices and at least $\binom{n-1}{2} + 2$ edges has a hamiltonian circuit [10, theor. 4.3].

PROOF. The proof is by induction on n. The statement is trivial for $n = 4$, so we proceed to the induction step. Let G be a graph with $n \geqq 5$ vertices and at least $\binom{n-1}{2} + r + 1$ edges. It is no loss of generality to assume that G is not complete. If we delete a vertex of G of degree at most $n-2$, then the resulting graph has $n-1$ vertices and at least $\binom{n-2}{2} + 2$ edges, and hence it has a circuit of length $n-2$, by the induction hypothesis and the remark preceding the proof. In particular, G has a path of length r. Now let P be any path of length r, say from vertex v_0 to vertex v_r. If $r = n-3$, then G has $\binom{n}{2} - 1$ edges and the theorem is true; so assume $r \leqq n-4$. If every vertex of G which is not on P has degree $n-1$, then P is clearly contained in a circuit of any length greater than $r+1$. Let u be a vertex of minimum degree among the vertices of G not on P. We assume $d(u) \leqq n-2$. Then $G-u$ has $n-1$ vertices and at least $\binom{n-2}{2}+r+1+(n-2-d(u))$ edges. By the induction hypothesis P is contained in circuits (in $G-u$) of lengths $r+2, r+3, \cdots, n-2$.

We shall now show that P is contained in a circuit of length $n-1$ in G. By the induction hypothesis this is the case provided $G-u \neq H_r$ and $G-u \neq H_{n-1,r}$. So assume $G-u=H_r$ or $G-u=H_{n-1,r}$. In particular $|E(G-u)|=\binom{n-2}{2}+r+1$, which implies that u has degree $n-2$ in G. By the minimality property of u, every vertex of $G-u$ not on P has degree at least $n-3$ in $G-u$. Consider first the case $G-u=H_r$, $n-1=r+5$. Every vertex of H_r has degree $r+4=n-2$ or $r+2=n-4$, so every vertex of $G-u$ not on P has degree $n-2$ in $G-u$. This clearly implies that P is contained in a circuit of length $n-1$. Consider next the case $G-u=H_{n-1,r}$. Every vertex of $H_{n-1,r}$ has degree $n-2$, $n-3$ or $r+1$. If the vertex of degree $r+1$ is contained in P, then P is clearly contained in a circuit of length $n-1$, so assume the opposite. Then $r+1 \geqq n-3$. If $r+1=n-2$, then $G-u$ is complete, so assume $r+1=n-3$. From this it follows that G is obtained from the complete graph with n vertices by deleting two edges, and P is a path in G of length $n-4$. It is now easy to see that P is contained in a circuit of length $n-1$.

We shall finally show that P is contained in a circuit of length n under the assumption that $G \neq H_r$ and $G \neq H_{n,r}$. We have already shown that P is in a circuit C of length $n-1$. Let Q be the path of length $n-r-1$ from v_0 to v_r, contained in C, but distinct from P. Denote Q by $x_0x_1\cdots x_{n-r-2}x_{n-r-1}$, where $x_0=v_0$ and $x_{n-r-1}=v_r$, and let u be the vertex of G not on C. Suppose u is adjacent to k of the vertices $\{x_1, x_2, \cdots, x_{n-r-2}\}$. Since $G \neq H_{n,r}$, we see that $d_G(u) \geqq r+2$, and thus $k \geqq 1$.

First, we show that we may restrict our attention to the cases $k=1$ and $k=2$. Suppose u is adjacent to x_i and x_j, $1 \leqq i < j \leqq n-r-2$. If $j=i+1$, then replacement in C of x_ix_{i+1} by x_iux_{i+1} yields the theorem. So assume $i+1<j$. If $x_{i-1}x_{j-1}$ is in $E(G)$ ($x_{i+1}x_{j+1}$ is in $E(G)$) replacement of $x_{i-1}x_i$ and $x_{j-1}x_j$ (respectively, x_ix_{i+1} and x_jx_{j+1}) in C by $x_{i-1}x_{j-1}$ and x_iux_j (respectively, $x_{i+1}x_{j+1}$ and x_iux_j) yields the theorem. So if u is adjacent to $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ (all on C), $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n-r-2$, then we may assume without loss of generality that $\{x_{i_1-1}, x_{i_2-1}, \cdots, x_{i_k-1}\}$ and $\{x_{i_1+1}, x_{i_2+1}, \cdots, x_{i_k+1}\}$ are two independent sets of vertices in G with at most $k-1$ common vertices. This accounts for at least

$2(k-1)+\binom{k-1}{2}$ edges in $\bar{G}$. Since u is nonadjacent to $n-r-2-k$ vertices of $\{x_1, x_2, \cdots, x_{n-r-2}\}$,

$$(1) \qquad |E(\bar{G})| \geqq 2(k-1)+\binom{k-1}{2}+n-r-2-k.$$

But $|E(\bar{G})| \leqq n-r-2$, so that $0 \geqq \binom{k}{2}-1$, or $k \leqq 2$. So we now treat the cases $k=1$ and $k=2$.

If $k=2$, the right side of (1) is $n-r-2$. This means that all edges of $\bar{G}$ have been determined in the count leading to (1). We deduce that u is adjacent to each of the vertices $v_0, v_1, \cdots, v_r$ and that x_1 is adjacent to x_{i_2+1}. Replacement in C of x_0x_1 and $x_{i_2}x_{i_2+1}$ by $x_{i_2+1}x_1$ and $x_{i_2}ux_0$ yields the theorem.

If $k=1$, then $r+2 \leqq d_G(u) \leqq (n-1)-(n-r-3)=r+2$. Hence, $|E(\bar{G}-u)| = |E(\bar{G})| - d_{\bar{G}}(u) \leqq (n-r-2)-(n-1-d_G(u))=1$. Moreover, u is adjacent to every vertex of P. Denote i_1 by i. If $i=1$ or $i=n-r-2$, then replacement yields the theorem. So we assume $2 \leqq i \leqq n-r-3$. Since $|E(\bar{G}-u)| \leqq 1$, not both x_1x_{i+1} and $x_{i-1}x_{n-r-2}$ are in $E(\bar{G})$ unless these two edges are actually the same edge, i.e., unless $i=2$ and $i+1=n-r-2$. In this case $n=r+5$ and thus, $G=H_r$, a contradiction. If x_1x_{i+1} is in $E(G)$ ($x_{i-1}x_{n-r-2}$ is in $E(G)$), then replacement in C of x_0x_1 and x_ix_{i+1} (respectively, $x_{i-1}x_i$ and $x_{n-r-2}x_{n-r-1}$) by x_1x_{i+1} and x_0ux_i (respectively, $x_{i-1}x_{n-r-2}$ and x_iux_{n-r-1}) yields a circuit of length n containing P.

In any case, P is contained in a circuit of length n so that the proof of the theorem is complete.

We point out that the case $r=1$ of the previous theorem was proved using different methods by B. Alspach and T. Brown [2] and recall that the case $s=n$ was partially treated by H. V. Kronk [7].

Corollary 2.4. *Let r and n be integers, $1 \leqq r \leqq n-2$.*

$$\text{Then } m_1(n; r, s) = \begin{cases} \binom{n-1}{2}+r+1, & \text{if } r+2 \leqq s \leqq n-1 \\ \binom{n-1}{2}+r+2, & \text{if } s=n \geqq r+3. \end{cases}$$

3. The function $m_2(n; 1, s)$

Definition 3.1. Let n and s be integers, $2 \leqq s \leqq n$. Let $n=(s-1)i+r$, $1 \leqq r \leqq s-1$. We denote by $F_{n,s}$ the multigraph obtained from a star with $i+1$

edges by replacing i edges by a circuit of length s and one edge by a circuit of length r (here a circuit of length 1 is understood to have one vertex and no edges), cf. Figure 1.

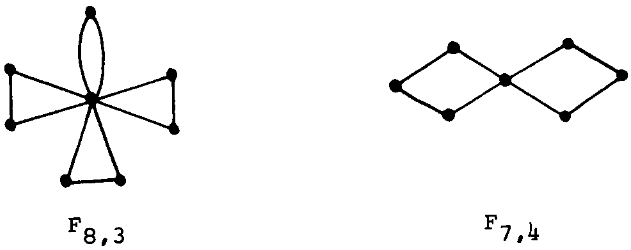


Fig. 1.

The multigraph $F_{n,s}$ is a graph, unless $s = 2$ or $r = 2$. Note that $F_{n,s}$ has property $p(1, \leqq s)$. $F_{n,s}$ has n vertices and the number of edges is

$$f(n, s) = \begin{cases} si \quad , & \text{if} \quad r = 1 \\ si + r, & \text{if} \quad r \neq 1 . \end{cases}$$

We study the function $f(n, s)$ in the next lemma.

LEMMA 3.2. *Let k, s, and n be integers $2 \leqq k \leqq s \leqq n$. Write $n = (s-1)i + r$, $1 \leqq r \leqq s-1$. Then*

i) $$f(n-k+1, s) = \begin{cases} f(n, s) - k + 1, & \text{if} \quad 2 \leqq k \leqq r-1 \\ f(n, s) - k, \quad , & \text{if} \quad r \leqq k \leqq s \end{cases}$$

ii) $$f(n, s) \leqq f(n, s-1) \quad \textit{with equality if and only if} \quad 1 < r \quad \textit{and} \quad r + i \leqq s - 1 .$$

PROOF. Part (i) follows from the observation that for $2 \leqq s \leqq m$,

$$f(m+1, s) - f(m, s) = \begin{cases} 2, & \text{if} \quad m \equiv 1 \pmod{s-1} \\ 1, & \text{if} \quad m \not\equiv 1 \pmod{s-1} . \end{cases}$$

The proof of part (ii) is straightforward and is omitted.

THEOREM 3.3. *Let s and n be integers, $2 \leqq s \leqq n$, and let G be a connected multigraph with n vertices. If G has property $p(1, \leqq s)$, then $|E(G)| \geqq f(n, s)$. Moreover, if $|E(G)| = f(n, s)$ and $n \equiv 1 \pmod{s-1}$ as well, then G is a connected graph each block of which is a circuit of length s.*

PROOF. The proof is by induction on n and s. Let G be a connected multigraph with n vertices which has property $p(1, \leqq s)$. For $2 = s \leqq n$, since G

is connected $|E(G)| \geqq 2(n-1) = f(n,2)$. If $|E(G)| = f(n,2)$, then it is easily seen that multigraph G is obtained from a tree by replacing each edge by a circuit of length 2. Also, for $2 \leqq s = n$, $|E(G)| \geqq n = f(n,n)$, since G is connected and must contain at least one circuit. If $2 \leqq s = n$ and $|E(G)| = n$, then G is exactly a circuit. Let $3 \leqq s < n$. Assume that for each $n' < n$ the theorem holds for all s', $2 \leqq s' < n'$. Also assume that the theorem holds for all s', $2 \leqq s' < s < n$.

If G contains no circuit of length s, then G has property $p(1, \leqq (s-1))$. By the induction hypothesis and part (ii) of the lemma, $|E(G)| \geqq f(n, s-1) \geqq f(n,s)$. If $|E(G)| = f(n,s)$, then $f(n,s-1) = f(n,s)$, and hence, by part (ii) of the lemma, $n \not\equiv 1 \pmod{s-1}$.

So, we may assume that G contains a circuit C of length $s < n$. Contract all edges of G which have both ends on C to a single vertex x, yielding a new multigraph H with $|V(H)| = n - s + 1$ and $|E(H)| \leqq |E(G)| - s$. H has property $p(1, \leqq s)$, so by our induction hypothesis, $|E(H)| \geqq f(n-s+1, s)$. By part (i) of the Lemma

$$|E(G)| \geqq |E(H)| + s \geqq f(n-s+1, s) + s = f(n,s). \tag{3}$$

If $|E(G)| = f(n,s)$ and $n \equiv 1 \pmod{s-1}$, then from (3) we see that $|E(H)| = f(n-s+1, s)$, $|V(H)| \equiv 1 \pmod{s-1}$, and that C has no diagonals, i.e., the only edges of G with both ends on C are exactly the edges of C. Clearly H is connected. By our induction hypothesis, each block of H is a circuit of length s. Consider a circuit C' of H containing x. In G the edges of C' induce either a circuit or a path connecting two distinct vertices of C. Let e be any edge of C'. G contains a circuit C'' including the edge e and of length at most s. Knowing the structure of H, we easily deduce that C'' contains all edges of C'. Hence, since C' has length s, $C'' = C'$, i.e., C' and C have precisely one vertex in common. From this it follows that G is a connected graph each block of which is a circuit of length s.

By induction the theorem follows.

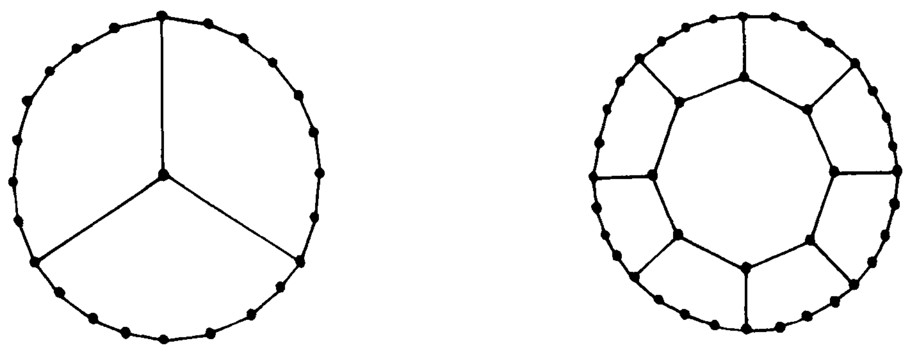
Corollary 3.4. *Let s and n be integers, $2 \leqq s \leqq n$. If $n = (s-1)i + r$, $1 \leqq r \leqq s-1$, then*

$$m_2(n;1,s) = f(n,s) = \begin{cases} si, & \text{if } r = 1 \\ si + r, & \text{if } r \neq 1. \end{cases}$$

Proof. The inequality $m_2(n;1,s) \geqq f(n,s)$ follows from the previous theorem. To show the reverse inequality take $F_{(s-1)i+1,s}$, select two vertices which are connected by a path of length $s-r$, and add (if $r>1$) a path of length r between these vertices.

The proof of the Corollary shows that the extremal graphs may contain blocks which are not circuits in the cases $n \not\equiv 1 \bmod (s-1)$. Also, there exist extremal graphs containing no two edge-disjoint circuits. For example, consider the graph of Figure 2a.

Of course, there are edge-minimal connected graphs which have property $p(1,s)$ and which are not edge-minimum. Consider for example the graph of Figure 2b.



(a) (b)

Fig. 2.

4. The function $m_2(n;2,4)$

Now we turn to the property $p(2, \leqq 4)$. It is easy to see that a connected graph G with n vertices, $n \geqq 3$, has property $p(2,3)$ if and only if $G = K_n$. However, there seems to be no simple characterization of graphs with property $p(2,4)$ (respectively, $p(2, \leqq 4)$). It can be shown that the only cubic graphs with $p(2, \leqq 4)$ are K_4, $K_{3,3}$, the 3-cube and the triangular based prism (the cartesian product of K_3 and K_2). Regular graphs with property $p(2, \leqq 4)$ include K_{n+1}, $K_{n,n}$, K_{n+2}-(edges in a perfect matching) (if n is even), and the cartesian product of K_n and K_2.

A lower bound on the number of edges in graphs with property $p(2, \leqq 4)$ is obtained in the next theorem.

THEOREM 4.1. *If G is a connected graph with n vertices, $n \geqq 4$, which has property $p(2, \leqq 4)$, then $|E(G)| \geqq 2n-4$.*

PROOF. Let G be a connected graph with n vertices, $n \geqq 4$, which has property $p(2, \leqq 4)$. Let H be a subgraph of G such that

i) $|E(H)| \geqq 2|V(H)| - 4$,

ii) for each $x \in V(H)$, $d_H(x) \geqq 2$, and

iii) $|V(H)|$ is maximum.

Such an H exists since G contains a circuit of length 4. If $|V(H)| = n$, then $|E(G)| \geqq |E(H)| \geqq 2n - 4$. Suppose $|V(H)| < n$. Let H' be the subgraph of G spanned by the vertices in $V(G) - V(H)$ which are adjacent to some vertex of H. $V(H') \neq \emptyset$ since G is connected and $|V(H)| < n$. If $v \in V(H')$ and $v \in N(x) \cap N(y)$ with $x \neq y$ in $V(H)$, then $H + v$ (the subgraph of G composed of H and v and all edges of G with one end in $V(H)$ and one end v) satisfies (i) and (ii), a contradiction to (iii). So if v is in $V(H')$ there is exactly one x in $V(H)$ with v adjacent to x. By (ii), let x_1 and x_2 be in $N_H(x)$. Path x_1xv is in a circuit C_1 of length 4, so from our previous remark, there is a v_1 in $V(H')$ so that C_1 is given by $x_1xvv_1x_1$. Similarly, x_2xv is in a circuit C_2 of length 4 given by $x_2xvv_2x_2$ with $v_2 \in V(H')$. Also $v_1 \neq v_2$, since $x_1 \neq x_2$. Thus, $d_{H'}(v) \geqq 2$, for every v in $V(H')$, and $|E(H')| \geqq |V(H')|$. Let H'' be the subgraph of G spanned by vertices in $V(H) \cup V(H')$. Then $|E(H'')| \geqq |E(H)| + |E(H')| + |V(H')| \geqq 2|V(H)| - 4 + 2|V(H')| = 2|V(H'')| - 4$, i.e., (i) is satisfied by H''. Since $d_{H''}(x) \geqq 2$ for every x in $V(H'')$, we have contradicted the choice of H. Consequently, $|V(H)| = n$ and the theorem follows.

In order to show that the bound given in the previous theorem is best possible, we merely need to consider $K_{2;n-2}$. So, we have determined $m_2(n; 2, 4)$.

COROLLARY 4.2. $m_2(n; 2, 4) = 2n - 4$ *for* $n \geqq 4$.

COROLLARY 4.3. *The only edge-minimum cubic graph with property* $p(2, \leqq 4)$ *is the* 3-*cube.*

For each $n \geqq 6$, we can find a non-complete bipartite edge-minimum graph with n vertices with property $p(2, \leqq 4)$ (the case $n = 8$ is given in Corollary 4.3). In fact, we now describe all the edge-minimum graphs with property $p(2, \leqq 4)$.

DEFINITION 4.4. Let B be the collection of graphs defined as follows:

i) $K_{2,2}$ is in B.

ii) If G is in B and x and y are two distinct nonadjacent vertices of G such that $N(x) = N(y)$, and z is a new vertex (i.e., not in $V(G)$), then the graph H with $V(H) = V(G) \cup \{z\}$ and $E(H) = E(G) \cup \{xz, yz\}$ is also in B.

It is easy to prove by induction on the number of vertices that every graph in B has property $p(2, 4)$.

All edge-minimum connected graphs with property $p(2, \leqq 4)$ are given in the next theorem.

THEOREM 4.5. *Let G be a connected graph with n vertices, $n \geqq 4$, which has property $p(2, \leqq 4)$. If $|E(G)| \leqq 2n-4$, then G has property $p(2,4)$, and G is the 3-cube or G is in B.*

We omit the proof of this theorem, as it is too long to be included here. It is conducted by induction on n, and depends strongly on the ideas in the proof of Theorem 4.1.

We mention another generalization of Theorem 4.1.

THEOREM 4.6. *Let k be an integer, $k \geqq 4, k \neq 6$. If G is a connected graph with n vertices, $n \geqq k$, which has property $p(k-2, \leqq k)$, then $|E(G)| \geqq 2n-k$.*

We also omit the proof of this theorem. Note that the theorem is best possible for $k = 5$ (as demonstrated by the graph obtained from $K_{2,n-3}$ by inserting a new vertex of degree 2 on one of its edges). Thus, $m_2(n; 3, 5) = 2n - 5$. It can easily be shown, using the method of Theorem 4.1, that every connected graph with n vertices which has property $p(4, \leqq 6)$ has at least $(3/2)n - 3$ edges. In the next section we show this is best possible, when n is even.

5. Bounds for $m_2(n; r, s)$

Edge-minimum graphs with property $p(2, s)$, $s > 4$, are apparently difficult to determine, particularly when s is odd. In this section we present some constructions which yield upper bounds on the number of edges in edge-minimum graphs with property $p(r, s)$, i.e., on $m_2(n; r, s)$. Values of $m_2(n; 1, s)$, $m_2(n; 2, 4)$ and $m_2(n, 3, 5)$, are given in previous sections.

Consider the graphs indicated in Figure 3.

It follows from Figure 3 that

a) $m_2(n; 2, s) \leqq (3/2)n$ when n and s are even integers $\geqq 6$.

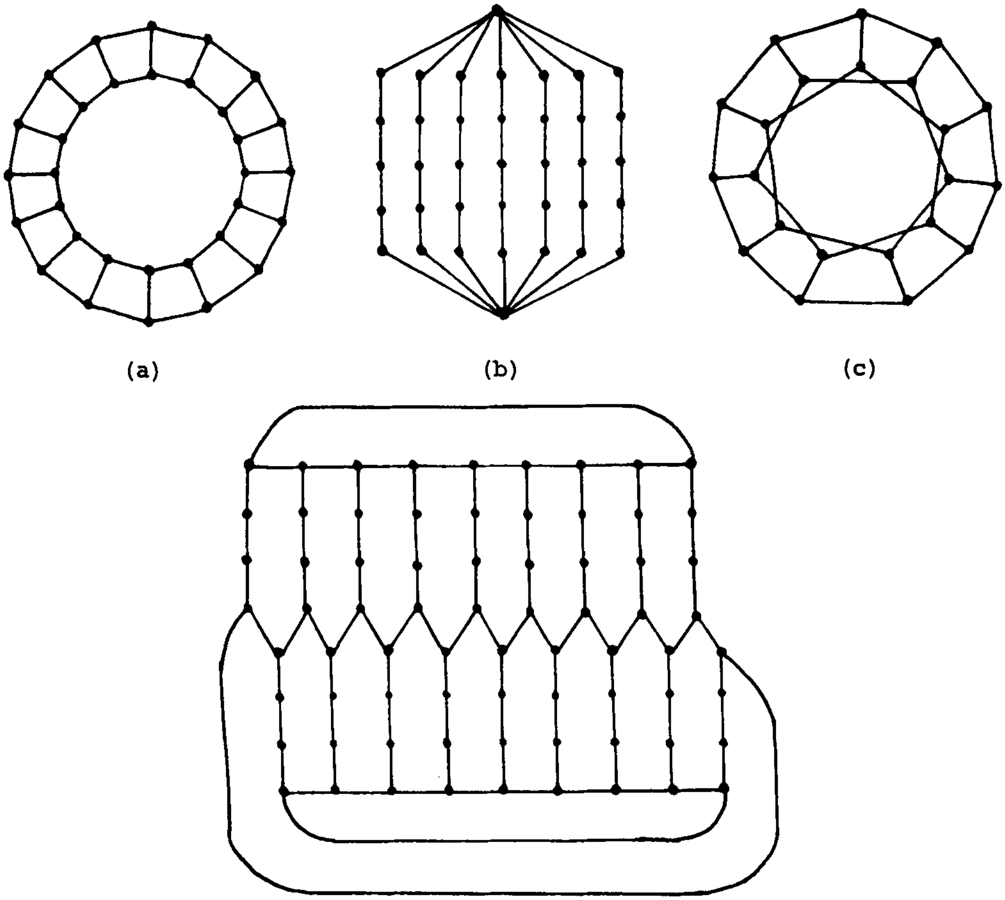
b) $m_2(n; r, s) \leqq (s/(s-2))(n-2)$ for $n = ((s-2)/s)i + 2$, $i \geqq 2$, $2 \leqq r \leqq (s/2) + 1$, and $s \geqq 6$, s even.

c) $m_2(2s; 2, s) \leqq m_2(2s; 3, s) \leqq 3s$ for s odd, $s \geqq 5$.

d) $m_2(s^2 - s; 2, s) \leqq s^2 + s$ for s odd, $s \geqq 7$.

Note that (c) shows that the inequality of (a) also holds, when s is odd, and $n = 2s$. Also note that the inequality of (b) reduces to that of (d), if we put $r = 2$ and $n = s^2 - s$.

Combining the last paragraph of Section 4 with (b), we see that $m_2(n; 4, 6) = (3/2)n - 3$, for n even.



(a) (b) (c) (d)

Fig. 3.

6. A generalization of property $p(2, s)$

DEFINITION 6.1. A graph G has property $q(2, s)$ (respectively, $q(2, \leqq s)$) if each pair of edges is contained in a circuit of length s (respectively, less than or equal to s). Let $\mu(n, s)$ (respectively, $\mu(n, \leqq s)$) denote the smallest integer such that there exists a connected graph with n vertices and $\mu(n, s)$ (respectively, $\mu(n, \leqq s)$) edges which has property $q(2, s)$ (respectively, $q(2, \leqq s)$). $\mu(n, s)$ is denoted $\mu_E(n, s)$ in [9].

Clearly, $\mu(n, \leqq s) \leqq \mu(n, s)$ and $m_2(n; 2, s) \leqq \mu(n, s)$. It is easy to see that the 3-cube does not have property $q(2, 4)$ and that the graphs $K_{2,n-2}$ are the only graphs in B (see Definition 4.4) which have property $q(2,4)$. So by Theorem 4.5, we obtain the following result of U. S. R. Murty [9, theor. 1^E]:

THEOREM 6.2. *$\mu(n,4)=2n-4$, and $K_{2,n-2}$ is the corresponding extremal graph.*

Murty also proved [9, theor. 2^E] that $\mu(n,6)=[(3n-5)/2]$. The graphs in Section 5 demonstrating the inequality $m_2(n;2,2s)\leqq(s/(s-1))(n-2)$ when $n\equiv 2 \bmod (s-1)$ also show $\mu(n,\leqq 2s)\leqq\mu(n,2s)\leqq(s/(s-1))(n-2)$ for these values of n. We shall show that $(n,\leqq 2s)=\{(s/(s-1))(n-2)\}$ for all $n\geqq 2s$. For this we need the following lemma.

LEMMA 6.3. *Let H be a graph with radius $r(H)=r$, and let P be a path of length s such that P and H have exactly the endvertices, v and w say, of P in common. Then*

a) $r(H\cup P)\geqq\frac{1}{2}(s+d_H(v,w)-1)$, *and*

b) $r(H\cup P)\geqq\min\{r,s\}$.

PROOF OF (a). Let H' denote the graph $H\cup P$ and let P' be a shortest path of H connecting v and w. P' has length $d=d_H(v,w)$. Consider first any vertex u on P but not in H. It is easy to see that the circuit $P\cup P'$ has radius $[\frac{1}{2}(s+d)]$, so there is a vertex u' in $P\cup P'$ such that $d_{P\cup P'}(u,u')\geqq\frac{1}{2}(s+d-1)$. But $d_{P\cup P'}(u,u')=d_{H'}(u,u')$, so $e_{H'}(u)\geqq\frac{1}{2}(s+d-1)$ ($e_{H'}(u)$ denotes the eccentricity of u in H'). Consider next a vertex u in H. Put $\alpha=d_H(u,v)$ and $\beta=d_H(u,w)$ and assume, without loss of generality, that $\alpha\leqq\beta$. Suppose first $\alpha+s\geqq\beta$. In this case let u' be the vertex of P whose distance in P from w is $t=[\frac{1}{2}(\alpha+s-\beta)]$. This vertex exists since $\alpha\leqq\beta$ implies $t\leqq s$. Then $d_{H'}(u,u')=\min\{\alpha+s-t,\beta+t\}$. By choice of t, $\alpha+s-t\geqq\beta+t=[\frac{1}{2}(\alpha+\beta+s)]\geqq\frac{1}{2}(d+s-1)$. So $e_{H'}(u)\geqq\frac{1}{2}(d+s-1)$. Suppose next $\alpha+s<\beta$. If $\alpha+s\geqq\frac{1}{2}(d+s-1)$, then $e_{H'}(u)\geqq d_{H'}(u,w)\geqq\frac{1}{2}(d+s-1)$, so assume $\alpha+s<\frac{1}{2}(d+s-1)$. Let u' denote the vertex on P' whose distance in P', i.e., in H, from w is $t=[\frac{1}{2}(d-s-2\alpha)]$. Then $d_H(u,u')\geqq d_H(u',v)-d_H(u,v)=d-t-\alpha\geqq\frac{1}{2}(d+s-1)$.

So $e_{H'}(u)\geqq d_{H'}(u,u')=\min\{d_H(u,u'),\alpha+s+t\}\geqq\frac{1}{2}(d+s-1)$. In each case $e_{H'}(u)\geqq\frac{1}{2}(d+s-1)$, so $r(H')\geqq\frac{1}{2}(d+s-1)$, and (a) is proved.

PROOF OF (b). We shall prove that for every vertex u in H', $e_{H'}(u)\geqq r$ or $e_{H'}(u)\geqq s$. Consider first a vertex $u\in V(H)$. There exists a u' in $V(H)$ such that $d_H(u,u')=e_H(u)\geqq r$. Either $d_{H'}(u,u')=d_H(u,u')\geqq r$ or else a shortest path in H' from u to u' includes P, in which case $d_{H'}(u,u')\geqq s$. So $e_{H'}(u)\geqq\min\{r,s\}$. Consider next a vertex u on P but not in H. Vertex u partitions P into a $v-u$

path of length t, say, and a $u-w$ path of length $s-t$. Let u' be the vertex on P' whose distance (in P') from v is $\min\{t, d_H(v,w)\}$. Let $z \in V(H)$ be a vertex such that $d_H(z,u') = e_H(u') \geqq r$. Then $d_H(z,v) \geqq d_H(z,u') - d_H(u',v) \geqq r-t$, and

$$d_H(z,w) \geqq \begin{cases} r-(d_H(v,w)-t), & \text{if} \quad t < d_H(v,w) \\ r \quad , & \text{if} \quad t \geqq d_H(v,w). \end{cases}$$

Hence,

$$\begin{aligned} e_{H'}(u) \geqq d_{H'}(u,z) &= \min\{t + d_H(z,v), \quad s-t+d_H(z,w)\} \\ &\geqq \begin{cases} \min\{r, r+s-d_H(v,w)\}, & \text{if} \quad t < d_H(v,w) \\ \min\{r, s-t+r\} \quad , & \text{if} \quad t \geqq d_H(v,w). \end{cases} \end{aligned}$$

If $d_H(v,w) \leqq s$, we have proved that $e_{H'}(u) \geqq r$, so assume $d_H(v,w) \geqq s+1$. Then $e_{H'}(u) \geqq e_{P \cup P'}(u) = [\frac{1}{2}(s + d_H(v,w))] \geqq [\frac{1}{2}(2s+1)] = s$. In either case $e_{H'}(u) \geqq \min\{r,s\}$ and (b) is proved.

Remark 6.4. The graph consisting of m paths, each of length k, and one path of length $r+1$, $1 \leqq r \leqq k-1$, such that each pair of paths have exactly the endvertices in common has $2+(k-1)m+r$ vertices, $km+r+1$ edges, and it has property $q(2, \leqq 2k)$.

Theorem 6.5. *Let G be a connected graph with $|V(G)| = n = 2+(k-1)m+r$, where $1 \leqq r \leqq k-1$. If G has property $q(2, \leqq 2k)$, then $|E(G)| \geqq km+r+1$.*

Proof. A non-empty, connected subgraph H of G spanned by $2+(k-1)m'+r'$ vertices $(1 \leqq r' \leqq k-1)$ of G is called *admissible* if

i) $|E(H)| \geqq km'+r'+1$, and

ii) $|E(H)| > km'+r'+1$, if $r(H) \leqq r'$.

It is easy to see that every subgraph spanned by the vertices of a circuit of length not exceeding $2k$ is admissible. Let H be a maximal admissible subgraph. We shall show that $H = G$. Suppose, therefore, that $H \neq G$.

We shall first consider the case where G contains a path P of length not exceeding k such that P and H have precisely the endvertices, v and w say, of P in common. Let s denote the length of P and let H' be the subgraph of G spanned by $V(H) \cup V(P)$. Then $|V(H')| = |V(H)| + s - 1 = 2+(k-1)m'+r'+s-1$, and $|E(H')| \geqq |E(H)| + s \geqq km'+r'+s+1$. If $r'+s-1 \leqq k-1$, then H' is an admissible subgraph, contradicting the maximality property of H. So we may assume $r'+s > k$. Then we write $|V(H')| = 2+(k-1)(m'+1)+r'+s-k$ and $|E(H')| \geqq k(m'+1)+r'+s-k+1$. If the last inequality is strict, then H' is admissible, so assume it is not strict. Hence, the inequality $|E(H')| \geqq |E(H)| + s$ is an equality, and this implies $|E(H)| = km' + r' + 1$. Since H is admissible, $r(H) \geqq r' + 1$. By Lemma 6.3(b), $r(H') \geqq \min\{r' + 1, s\}$. Since $r' \leqq k - 1$ and $s \leqq k$, it follows that $r(H') \geqq \min\{r' + 1, s\} \geqq r' + s - k + 1$. Hence H' is admissible, a contradiction.

We next consider the case where no such path P exists. Let e be an edge of G incident with vertices x in $V(H)$ and y in $V(G) - V(H)$. Let z be a vertex of maximum distance (in H) from x. Let e' denote any edge of H incident with z. Let C be a shortest circuit in G containing e and e'. Since G has property $q(2, \leqq 2k)$, C has length not exceeding $2k$. There is a unique decomposition of C into paths, $P^1, P^2, \cdots, P^q, P^{q+1}, \cdots, P^{q'}$ such that each of P^i $(1 \leqq i \leqq q)$ has exactly its endvertices in common with H, each of P^j $(q + 1 \leqq j \leqq q')$ is contained in H, and $P^i \cap P^j = \emptyset$ for $q < i < j \leqq q'$. By assumption, each of the paths P^i $(1 \leqq i \leqq q)$ has length greater than k. But this implies that $q = 1$ and $q' = 2$. Let v denote the endvertex of P^1 other than x, and let s denote the length of P^1 $(k + 1 \leqq s < 2k)$. Let H' denote the subgraph of G spanned by $V(H) \cup V(P^1)$. Then $|V(H')| = 2 + (k - 1)(m' + 1) + r' + s - k$, and $|E(H')| \geqq |E(H)| + s \geqq k(m' + 1) + r' + s - k + 1$. Suppose first that $|E(H)| = km' + r' + 1$. Then $r(H) \geqq r' + 1$, which implies that P^2 has length at least $r' + 1$, because it contains an $x - z$ path. Then s, the length of P^1, is at most $2k - r' - 1$, which implies $r' + s - k \leqq k - 1$. To show that H' is admissible, and thus obtain a contradiction, we need only show that $r(H') \geqq r' + s - k + 1$. The $x - z$ path contained in P^2 has length at least $r' + 1$. Hence, the $z - v$ path contained in P^2 has length at most $2k - s - r' - 1$, which implies $d_H(v, x) \geqq d_H(x, z) - d_H(z, v) \geqq (r' + 1) - (2k - s - r' - 1) = 2r' + s - 2k + 2$. By Lemma 6.3(a), $r(H') \geqq \frac{1}{2}(s + (2r' + s - 2k + 2) - 1) = r' + s - k + 1 - \frac{1}{2}$. So H' is admissible, a contradiction. Suppose next that $|E(H)| > km' + r' + 1$. If $r' + s - k \leqq k - 1$, then we see from the inequality $|E(H')| \geqq |E(H)| + s > k(m' + 1) + r' + s - k + 1$ that H' is admissible. So we may assume that $r' + s - k \geqq k$. Then we write $|V(H')| = 2 + (k - 1)(m' + 2) + r' + s - 2k + 1$ and $|E(H')| \geqq k(m' + 2) + r' + s - 2k + 2$. Furthermore, by Lemma 6.3(a), $r(H') \geqq \frac{1}{2}(s + d_H(x, v) - 1) \geqq \frac{1}{2}s > r' + s - 2k + 1$, because $r' \leqq k - 1$ and $s < 2k$. This shows that H' is admissible, a contradiction which proves the theorem.

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